MATHEMATICS

DIFFERENTIATION IN NON-ARCHIMEDEAN VALUED FIELDS

BY

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Introduction

Let K be a complete non-archimedean valued field. The definition of the derivative of a real or complex function can without difficulties be translated for functions $f: K \to K$. In § 2 we give a general definition and derive some properties, the most important of which seems to be the local invertibility theorem (2.3). A striking difference with the "classical situation" is that the differential equation $y' = 0$ has many solutions (even homeomorphisms (2.7)), which seems to indicate that there is no hope for a reasonable translation of the mean value theorem and the Taylor expansion. In § 3 and § 4 we restrict ourselves to spaces K^n where K is a local field. We define an equivalence relation in the class Ω of compact open subsets of K^n by "$U \sim V$ iff there exists a diffeomorphism (i.e., a homeomorphism σ such that σ and σ^{-1} are differentiable) of U onto V".

It turns out that there are exactly $q-1$ equivalence classes where q is the order of the residue class field of K. Moreover, if $U \sim V$ one can choose σ to be of a very simple sort namely a so-called locally linear function ((3.5) and (3.6)). For analytic functions the above result was obtained by J-P. SERRE [5]. In § 4 we investigate an L-valued Haar measure on K^n, where L is a suitable non-archimedean field and define a notion of "derivative in measure". Theorem (4.5) establishes a relationship with the derivative in the usual sense. A substitution theorem for integrals is obtained. A surprising fact is that one can construct an example of a homeomorphism $\sigma: \mathbf{Q}_p \times \mathbf{Q}_p \to \mathbf{Q}_p$ that keeps the measure invariant. (see (4.3)). As a consequence an n-dimensional integral of f can be written as a one-dimensional integral of the transformed function $f \circ \sigma$.

§ 1. *Preliminaries*

In the sequel $\mathbf{N}$ stands for the set of the positive integers, $\mathbf{Z}$ for the set of the integers and $\mathbf{R}$ for the set of the real numbers. By K is always

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meant a complete non-archimedean valued field, the valuation is supposed to be non-trivial. The residue class field of K is denoted by k. K is called a local field, if K is locally compact. The p-adic number field is denoted by $\mathbf{Q}_p$ and the p-adic integers by $\mathbf{Z}_p$. By K^* we mean the (topological) multiplicative group of K, and for a field L we write $\chi(L)$ for the characteristic of L. A normed space over K is a K-vector space B, together with a function $\| \ \|: B \to \mathbf{R}$ satisfying

$$\|x\| \geqslant 0; \ \|x\| = 0 \text{ if and only if } x = 0, \tag{1}$$

$$\|\lambda x\| = |\lambda| \ \|x\|, \tag{2}$$

$$\|x+y\| \leqslant \max (\|x\|, \|y\|) \tag{3}$$

for all $x, y \in B$, $\lambda \in K$. A set $\{x_1, \ldots, x_n\} \subset B$ is called (norm-) orthogonal if for every $\xi_1, \ldots, \xi_n \in K$ we have

$$\| \sum_{i=1}^{n} \xi_i x_i \| = \max_i |\xi_i| \ \|x_i\|.$$

B is called a Banach space if B is complete. A ball with centre a in a normed space B is a subset of the form $\{x \in B: \|x-a\| \leqslant r\}$ or $\{x \in B: \|x-a\| < r\}$ where $r > 0$.

Every ball is closed and open, every point of a ball can be taken as its centre. Finite-dimensional normed spaces are always Banach spaces and all norms are equivalent. By K^n $(n \in \mathbf{N})$ we mean the normed K-linear space of n-tuples $x = (\xi_1, \ldots, \xi_n)$ ($\xi_i \in K$ for all i), where $\|x\| = \max |\xi_i|$. The canonical basis is denoted by $e_1, \ldots, e_n$, and forms obviously an orthogonal set. If B, B' are normed spaces over K and $A: B \to B'$ is a continuous linear map then we write

$$\|A\| = \sup \left\{ \frac{\|Ax\|}{\|x\|} : x \in B, \ x \neq 0 \right\}.$$

A is called an isomorphism if A is a bijection and A^{-1} is continuous. An isometry σ: $B \to B'$ is a map satisfying $\|\sigma(x) - \sigma(y)\| = \|x-y\|$ for all $x, y \in B$. Note that an isometry need not have nice additive or homogeneous properties. Let X be a topological space. $C(X \to K)$ denotes the space of the bounded continuous functions of X into K, $C_\infty(X \to K)$ is the subspace of $C(X \to K)$ consisting of all functions that vanish at infinity. Both spaces are equipped with the supremum norm.

Lemma (1.1). Let $O: K^n \to K^n$ be a linear mapping. Then the following are equivalent.

(a) O is an isometry.

(b) $|\det O| = 1$. For every matrix element a_{ij} of O we have $|a_{ij}| \leqslant 1$.

Lemma (1.2). Let K be a local field and let $A: K^n \to K^n$ be a linear bijection. Then A can be written as ODA', where O is a linear isometry, D is a diagonal map (i.e., the matrix of D with respect to the canonical basis is a diagonal matrix) and where $|\det A'| = 1$ and A' maps the unit ball E onto E.

The proofs are elementary and are left to the reader. For obtaining a proof of Lemma (1.2) one can use either a non-archimedean Gram-Schmidt process (see [3]), or the results of [7], section 2.

Lemma (1.3). Let B be a Banach space over K and let $P: B \to B$ be a linear isomorphism. If $Q: B \to B$ is continuous and linear and $\|P-Q\| < \|P^{-1}\|^{-1}$ then (Q is an isomorphism and) $P(S) = Q(S)$ for every ball S with centre 0.

Proof. Let I denote the identity operator. We have

$$\|I - P^{-1} Q\| = \|P^{-1}(P-Q)\| < 1. \text{ Hence } Q^{-1} P = \sum_{i=0}^{\infty} (I - P^{-1} Q)^i.$$

From the strong triangle inequality it follows that for all $x \in B$

$$\|P^{-1} Q(x)\| = \|Q^{-1} Px\| = \|x\|.$$

Hence, for every ball S with centre 0 we obtain $P^{-1}Q(S) = S$, or $Q(S) = P(S)$.

§ 2. *Differentiation*

Definition (2.1). Let B, B' be Banach spaces over K. Let a be an interior point of $U \subset B$. A function $f: U \to B'$ is called differentiable at a if there exist a continuous linear mapping $l_a: B \to B'$ (called the derivative of f in a) and a function $\varepsilon_a: U - a \to B'$ such that for all $h \in B$ for which $a + h \in U$

$$f(a+h) = f(a) + l_a(h) + \varepsilon_a(h),$$

where $\lim_{h \to 0} \|\varepsilon_a(h)\| \, \|h\|^{-1} = 0$.

Remark (2.2). The above definition is directly derived from the classical case. The derivative l_a is uniquely determined.

If f is differentiable at a then f is continuous at a.

One can formulate and prove a chain-rule for differentiable maps.

If, in addition, B' is a commutative Banach algebra, the usual product law holds. If B and B' are finite-dimensional, the mapping l_a can be represented (with respect to a certain basis) as the functional matrix. In the case $B = B' = K$ differentiability of f at a is equivalent to the existence of $\lim h^{-1}(f(a+h) - f(a))$. The proofs are only slight modifications of the well-known classical proofs and are omitted.

Theorem (2.3). (Local invertibility). Let B be a Banach space over K. Let a be an interior point of $U \subset B$, and let $f\colon U \to B$ be differentiable at a and continuous in a neighborhood of a. Let the derivative l_a be an isomorphism. Then for sufficiently small balls S with centre 0, f sends $a+S$ homeomorphically onto $f(a)+l_a(S)$.

Proof. Without loss of generality we may assume that U is a ball and that f is continuous in U. Write $f(a+h)=f(a)+l_a(h)+\varepsilon_a(h)$. Let S be any ball with centre 0 contained in $\{h \in B\colon \|\varepsilon_a(h)\| \leqslant \theta \|l_a{}^{-1}\|^{-1}\|h\|\}$, where $0<\theta<1$ arbitrary.

It is sufficient to show that the function $h \mapsto l_a(h)+\varepsilon_a(h)$ maps S homeomorphically onto $l_a(S)$, or equivalently, that $\xi\colon h \mapsto h+\eta(h)$, where $\eta = l_a{}^{-1} \circ \varepsilon_a$, is a homeomorphism of S onto S. Now for $h \in S$ we have $\|\eta(h)\| = \|l_a{}^{-1}\varepsilon_a(h)\| \leqslant \theta\|h\|$, hence $\|\xi(h)\| = \|h+\eta(h)\| = \max(\|h\|, \|\eta(h)\|) = \|h\|$, which implies that ξ maps S into S. Since f is continuous, ξ is continuous.

The series

$$\sum_{i=0}^{\infty} (-1)^i \eta^i(h)$$

converges uniformly on S and therefore represents a continuous function ϱ on S. An immediate verification shows that $\varrho \circ \xi = \xi \circ \varrho = id_S$.

Corollary (2.4). Let B be a Banach space over K and let $U \subset B$ be an open subset. Let $f\colon U \to B$ be differentiable in U such that for every $a \in U$ the derivative l_a is an isomorphism. Then f is an open mapping.

The above corollary generalizes a result of U. Güntzer [1], who obtained the statement for $B=K$, K algebraically closed, and where f can be represented as a Laurent series. If f has a continuous derivative we can prove a slightly stronger theorem.

Theorem (2.5). Let B, U, f be as in Corollary (2.4). Furthermore, suppose that the derivative $u \mapsto l_u$ $(u \in U)$ is continuous. Then for every $a \in U$ there is a neighborhood V of a such that for every $b \in V$, f maps $b+S$ homeomorphically onto $f(b)+l_a(S)$, where S is a sufficiently small ball with centre 0.

Proof. In virtue of Theorem (2.3) it is sufficient to show that if b is sufficiently close to a, then $l_a(S)=l_b(S)$ for all balls S with centre 0. But this follows from the continuity of the derivative and Lemma (1.3).

Theorem (2.6). Let B be a Banach space over K, let $U, V \subset B$ be open sets and let $\sigma\colon U \to V$ be a homeomorphism. Suppose that σ is differentiable at a and that the derivative l_a is an isomorphism. Then $\sigma^{-1}\colon V \to U$ is differentiable at $\sigma(a)$ with derivative $l_a{}^{-1}$.

Proof. Write $b+k=\sigma(a+h)$, $b=\sigma(a)$ and

$$\sigma(a+h)=\sigma(a)+l_a(h)+\varepsilon(h)\ldots(1)$$

$$\sigma^{-1}(b+k)=\sigma^{-1}(b)+l_a{}^{-1}(k)+\bar{\varepsilon}(k)\ldots(2).$$

From Theorem (2.3) we infer that for sufficiently small h we can find h' with $\|h'\| = \|h\|$ such that $\sigma(a+h) = \sigma(a) + l_a(h')$.

Hence we can rewrite (1) and (2) as: $k = l_a(h')$ and $h = l_a^{-1}(k) + \bar{\varepsilon}(k)$. So we have $\|h\| = \|h'\| = \|l_a^{-1}(k)\|$. Hence $\|\bar{\varepsilon}(k)\| \leqslant \max(\|h\|, \|l_a^{-1}(k)\|) \leqslant \|l_a^{-1}\| \, \|k\|$ for sufficiently small k, which means that $\|\bar{\varepsilon}(k)\| \, \|k\|^{-1}$ is bounded in a neighborhood of a. To show that the limit is 0, we apply σ to (2), and get

$$\bar{\varepsilon}(k) = -l_a^{-1} \circ \varepsilon(l_a^{-1}(k) + \bar{\varepsilon}(k)).$$

If we use the facts that $\|\varepsilon(h)\| \, \|h\|^{-1} \to 0$, that $\|\bar{\varepsilon}(k)\| \, \|k\|^{-1}$ is bounded and that $l_a^{-1}(k) + \bar{\varepsilon}(k) \to 0$, the rest is straightforward.

Remark (2.7). If in Theorem (2.3), Corollary (2.4), and Theorem (2.5) we omit the condition that l_a be an isomorphism, the proofs fail, and in fact the existence of abundantly many locally constant functions (i.e., functions that are constant on sets which are open and closed) shows that the theorems in this form cannot be true. On the other hand, one can construct homeomorphisms with everywhere vanishing derivatives as the following examples show.

1. (Borrowed from [2]). Every $x \in \mathbf{Z}_p$ can uniquely be written as a convergent sum $\sum_{i=0}^{\infty} a_i p^i$ where $a_i \in \{0, 1, \ldots, p-1\}$. Define $\varphi: \mathbf{Z}_p \to \mathbf{Z}_p$ as

$$\varphi(\sum a_i p^i) = \sum a_i p^{2i}.$$

It is easy to see that $|\varphi(x) - \varphi(y)| = |x-y|^2$ for all $x, y \in \mathbf{Z}_p$, which implies that $\varphi' = 0$ on $\mathbf{Z}_p$. φ is a homeomorphism of $\mathbf{Z}_p$ onto $\varphi(\mathbf{Z}_p) \neq \mathbf{Z}_p$. (One can prove that a homeomorphism of $\mathbf{Z}_p$ onto $\mathbf{Z}_p$ which has vanishing derivative does not exist.)

2. Let $\chi(K) = p \neq 0$ and suppose that the equation $x^p - \lambda = 0$ has a root for all $\lambda \in K$. The map $\sigma: x \mapsto x^p$ is a homeomorphism of K onto K and $\sigma' = 0$ everywhere.

§ 3. *Diffeomorphisms. Classification of Compact Open Subsets*

From now on we suppose that K is a local field. Let $\Omega(K^n)$ denote the class of all compact open subsets of $K^n (n \in \mathbf{N})$.

Definition (3.1). Let $U, V \in \Omega(K^n)$. A map $\sigma: U \to V$ is called a diffeomorphism if σ is a homeomorphism such that σ and σ^{-1} are differentiable. U and V are called diffeomorphic.

In order that $\sigma: U \to V$ be a diffeomorphism it is necessary and sufficient that σ be differentiable and injective, that $\sigma(U)$ be dense in V and that the Jacobian be nonzero everywhere. This follows from Corollary (2.4) and Theorem (2.6). In the sequel let q be the number of elements of the residue class field k of K, let $\chi(k) = p$, and let π be the greatest element in the value group of K that is smaller than 1. We normalize the (real) Haar measure μ on K^n by taking $\mu(E)=1$, where E is the unit ball. Since every ball then has a measure which is a power of q and since every $U \in \Omega(K^n)$ is the disjoint union of finitely many balls, $\mu(U)$ has the form aq^s where $a \in \mathbf{N}$, $s \in \mathbf{Z}$.

Hence the definition $\bar{\mu}(U)=\mu(U) \bmod (q-1)$ makes sense. $\bar{\mu}$ is an additive function on $\Omega(K^n)$ with values in $\mathbf{Z}/(q-1)$.

Definition (3.2). Let $U, V \in \Omega(K^n)$. We call U and V of the same type ($U \sim V$) if one of the following (equivalent) conditions is satisfied.
(i) $\bar{\mu}(U)=\bar{\mu}(V)$.
(ii) If U (resp. V) is a disjoint union of a (resp. b) balls, then $a=b \bmod (q-1)$.

Lemma (3.3). Let $A: K^n \to K^n$ be a linear isomorphism. Then for each $U \in \Omega(K^n)$ we have

$$\mu(AU)=\mu(U)|\det A|^{\alpha} \text{ where } \alpha=-\,{}^{\pi}\log q.$$

Proof. Clearly $U \mapsto \mu(AU)$ and $U \mapsto \mu(U)$ are nonzero invariant measures on $\Omega(K^n)$. Hence their ratio $\lambda(A)$ is independent of U. If B is also a linear isomorphism we have $\lambda(AB)=\lambda(A)\lambda(B)$, as can easily be verified. Write $A=ODA'$ as in Lemma (1.2). It follows immediately that $\lambda(O)=\lambda(A')=|\det O|=|\det A'|=1$.

What remains to be shown is that $\lambda(D)=|\det D|^{\alpha}$, in other words, that for $\lambda_i \in K$, $\lambda_i \neq 0$ $(i=1, 2, \ldots, n)$, we have $\mu(\{(\xi_1, \ldots, \xi_n) \in K^n : |\xi_i| \leqslant |\lambda_i|$ for all $i\})=|\Pi\lambda_i|^{\alpha}$. This can be shown, using the fact that μ is a product measure and that $\mu(\{\xi \in K : |\xi| \leqslant \pi^s\})=q^{-s}$ for $s \in \mathbf{Z}$.

Corollary (3.4). A linear isomorphism $A: K^n \to K^n$ does not change the type.

Proof. $|\det A|$ is an integral power of π. Hence $|\det A|^{\alpha}$ is an integral power of q and consequently $|\det A|=1 \bmod (q-1)$. Now apply Lemma (3.3).

Definition (3.5). Let B be a normed space over K and let $U \subset B$ be an open set. A map $\sigma: U \to B$ is called locally linear if each point of U has a neighborhood V such that for all $x \in V$: $\sigma(x)=a+Ax$ for some $a \in B$ and some linear map $A: B \to B$.

Theorem (3.6). Let $U, V \in \Omega(K^n)$. The following conditions are equivalent.
(a) U and V are of the same type.
(b) There is a differentiable homeomorphism of U onto V, for which the functional determinant vanishes nowhere.
(c) There is a locally linear homeomorphism of U onto V.

Proof. Clearly (c) $\to$ (b). To show (b) $\to$ (a), let $\sigma: U \to V$ satisfy (b). For every $a \in U$ there is a ball S_a such that σ maps $a+S_a$ homeomorphically onto $\sigma(a)+l_a(S_a)$. (Theorem (2.3)). The sets $a+S_a$ cover U, hence there are $a_1, \ldots, a_t \in U$ and balls $S_1, \ldots, S_t$ such that $\{a_i+S_i\}$ is a disjoint subcovering. By Corollary (3.4) the type of a_i+S_i is equal to the type of $\sigma(a_i+S_i)$. Summation over i yields $\bar{\mu}(U)=\bar{\mu}(V)$.

To show (a) $\to$ (c) we first observe that we can write U and V as a disjoint union of the same number of sets of the form

$$D_{a,r}=\{(\xi_1, \ldots, \xi_n) \in K^n: |\xi_i-a_i| \leqslant r_i\},$$

where r_i are positive elements of $|K|$, $r=(r_1, \ldots, r_n)$ and $a=(a_1, \ldots, a_n) \in K^n$. So it is sufficient to establish a locally linear map of, say, $D_{a,r}$ onto $D_{a',r'}$, and this is easily done by a translation over $-a$, followed by a suitable diagonal map, followed by a translation over a'.

§ 4. *Derivative in Measure. Substitution Theorem for Integrals*

Throughout this section K is a local field. Let $\Omega(K^n)$, π, k, p, q be as in § 3. For every non-trivial non-archimedean complete valued field L for which $\chi(l) \neq p$ there exists a nonzero continuous L-valued translation invariant functional m on $C_\infty(K^n \to L)$. (See [4]). For $m(f)$ we shall write $\int f(x)dx$, and for $U \in \Omega(K^n)$ the L-valued characteristic function of U is denoted by ε_U. By $\int_U f(x)dx$ we will mean $m(f\varepsilon_U)$ and $m(U)$ is an abbreviation for $m(\varepsilon_U)$. We normalize m by requiring that $m(E)=1$ where E is the unit ball. It follows that $|m(H)|=1$ for all compact open subgroups of K^n. For this and other elementary properties of m see [6]. In the sequel we shall fix L and $n, s \in \mathbf{N}$.

Definition (4.1). Let $\theta: \Omega(K^n) \to L$, let $b \in K^n$ and $\beta \in L$.

Suppose that for every $\varepsilon>0$ there exists a neighborhood U of b such that for all $V \in \Omega(K^n)$, $V \subset U$, $|m(V)|=1$ we have

$$|\theta(V)-\beta|<\varepsilon.$$

Then we write $\operatorname*{LIM}_{U \to b} \theta(U)=\beta$.

Definition (4.2). Let $U \subset K^n$ be an open set and let $\sigma: U \to K^s$ be an open continuous mapping. We say that σ is differentiable in measure (or m-differentiable) at $a \in U$ if an element $D_\sigma(a) \in L$ exists such that

$$\operatorname*{LIM}_{V \to a} m(V)^{-1} m(\sigma(V))=D_\sigma(a).$$

Examples (4.3). For an arbitrary isometry σ of K^n it is clear that $D_\sigma=1$. Such a σ need not be differentiable in the usual sense. Many differentiable functions are m-differentiable (4.5).

That the distinction between n and s, made in (4.2), makes sense is shown by the map $\sigma: \mathbf{Z}_p \to \mathbf{Z}_p \times \mathbf{Z}_p$ defined by

$$\sigma\left(\sum_0^\infty a_i p^i \right) = \left(\sum_0^\infty a_{2i} p^i, \sum_0^\infty a_{2i+1} p^i \right).$$

(Notations as in Remark (2.7)). Elementary computations show that σ is a homeomorphism, and that σ keeps the measure invariant, hence $D_\sigma = 1$. One can use this idea for the construction of similar homeomorphisms: $K^n \to K^s$.

Let $\lambda \in K^*$. The number $|\lambda|^{-^q\log q}$ is an integral power of q, hence it can be considered as an element of L. When we do so, we write $[\lambda]_L$ for this number. Since $\chi(L) \neq p$ we have $[\lambda]_L \neq 0$. Hence $[\quad]_L$ is a homomorphism of K^* into L^*, and since it is locally constant it is continuous. Furthermore $|[\lambda]_L| = 1$ for all $\lambda \in K^*$.

Lemma (4.4). Let $A: K^n \to K^n$ be an isomorphism. Then for all $U \in \Omega(K^n)$ we have

$$m(AU) = m(U)[\det A]_L.$$

Proof. The proof of Lemma (3.3) applies with only slight modifications.

Theorem (4.5). Let $U \subset K^n$ be open and let $\sigma: U \to K^n$ be differentiable with continuous derivative $x \mapsto l_x$ for which $\det l_x \neq 0$ for all $x \in U$. Then D_σ exists in U and for $a \in U$:

$$D_\sigma(a) = [\det l_a]_L.$$

Proof. By Corollary (2.4) σ is open. Hence Definition (4.2) applies to σ. Let $a \in U$. By Theorem (2.5) there is a neighborhood V of a, $V \subset U$ such that for each $v \in V$ and for sufficiently small balls S (with centre 0) σ maps $v+S$ homeomorphically onto $\sigma(v) + l_a(S)$. By Lemma (4.4) we have

$$m(\sigma(v+S)) = m(l_a(S)) = [\det l_a]_L \, m(v+S) \dots (1).$$

One can write every $W \subset V$, $W \in \Omega(K^n)$ as a finite disjoint union of such sets $v+S$ for which (1) holds. Summation gives us:

$$m(\sigma(W)) = [\det l_a]_L m(W)$$

for all $W \in \Omega(K^n)$, $W \subset V$. Hence a fortiori $D_\sigma(a) = [\det l_a]_L$.

Theorem (4.6). Let $U \subset K^n$ be open and let $\sigma: U \to K^s$ be m-differentiable in U. Then D_σ is continuous, $|D_\sigma| \leqslant 1$.

Proof. Since $|m(V)| \leqslant 1$ for all $V \in \Omega(K^n)$ it is clear that $|D_\sigma| \leqslant 1$.

Let $a \in U$ and $\varepsilon > 0$. There is a neighborhood V of a such that for all $W \subset V$, $W \in \Omega(K^n)$, $|m(W)| = 1$ we have:

$$|m(W)^{-1} m(\sigma(W)) - D_\sigma(a)| < \varepsilon \ldots (1).$$

Let $b \in V$. There is a neighborhood W' of b, $W' \in \Omega(K^n)$, $|m(W')| = 1$, $W' \subset V$ such that

$$|m(W')^{-1} m(\sigma(W')) - D_\sigma(b)| < \varepsilon \ldots (2).$$

Taking $W = W'$ in (1), we get

$$|D_\sigma(a) - D_\sigma(b)| \leqslant$$
$$\leqslant \max\,(|D_\sigma(a) - m(W')^{-1}\, m(\sigma(W'))|, |D_\sigma(b) - m(W')^{-1}\, m(\sigma(W'))|) < \varepsilon.$$

Theorem (4.7). Let $f: K^n \to L$ be continuous. For each $a \in K^n$ we have

$$\operatorname*{LIM}_{U \to a}\; m(U)^{-1} \int_U f(x)dx = f(a).$$

Proof. Let $\varepsilon > 0$. Choose a neighborhood U of a such that for $x \in U$ we have $|f(x) - f(a)| < \varepsilon$. For each $V \subset U$, $|m(V)| = 1$, $V \in \Omega(K^n)$ we have

$$|m(V)^{-1} \int_V f(x)dx - f(a)| = |\int_V (f(x) - f(a))dx| \leqslant \sup_{x \in V} |f(x) - f(a)| < \varepsilon.$$

Theorem (4.8). (Substitution theorem). Let $U \in \Omega(K^n)$, $V \in \Omega(K^s)$ and let $\sigma: U \to V$ be a homeomorphism. Then the following are equivalent.
(a) σ is m-differentiable on U.
(b) There is a continuous $\varphi: U \to L$ such that for all continuous $f: V \to L$:

$$\int_V f(x)dx = \int_U (f \circ \sigma)(t)\varphi(t)dt \ldots (1).$$

In addition, φ is uniquely determined and equal to D_σ. In case σ is continuously differentiable with nowhere vanishing Jacobian j, then $\varphi = [j]$.

Proof. (a) $\to$ (b): choose $\varphi = D_\sigma$. Both expressions in (1) now make sense and as functions of f they are continuous L-linear forms on $C(V \to L)$. Hence it is sufficient to check (1) only for functions of the type $\varepsilon_{\sigma(T)}$, where $T \subset U$, $T \in \Omega(K^n)$, $|m(T)| = 1$. The left hand expression becomes $m(\sigma(T))$. At the right side we get $\int_T D_\sigma(t)dt$. Let $\varepsilon > 0$.

For each $t \in T$ there is a ball S_t with centre t such that for all $y \in S_t$

$$|D_\sigma(y) - m(S_t)^{-1}\, m(\sigma(S_t))| < \varepsilon.$$

T is the disjoint union of finitely many of such balls $S_1, \ldots, S_r$. Define a function ψ on T by

$$\psi(y) = m(S_i)^{-1}\, m(\sigma(S_i)) \text{ when } y \in S_i.$$

It is clear that $\|D_\sigma - \psi\| < \varepsilon$, hence

$$\left| \int_T D_\sigma(t)dt - \int_T \psi(t)dt \right| < \varepsilon.$$

But $\int_T \psi(t)dt = \sum m(\sigma(S_i)) = m(\sigma(T))$.

(b) $\to$ (a): for each $T \in \Omega(K^n)$, $T \subset V$, we have

$$(\text{choose } f = \varepsilon_{\sigma(T)})\colon m(T)^{-1}\, m(\sigma(T)) = m(T)^{-1} \int_T \varphi(t)dt.$$

By Theorem (4.7) we have $\varphi = D_\sigma$. The last assertion follows from Theorem (4.5).

Corollary (4.9). Let $U \in \Omega(K^n)$, $V \in \Omega(K^s)$ and let $\sigma\colon U \to V$ be a surjective map that is locally a homeomorphism. Suppose that D_σ exists. For $t \in V$, define $n_\sigma(t)$ (regarded as an element of L) to be the number of $x \in U$ such that $\sigma(x) = t$. Then, for each continuous $f\colon V \to L$ we have

$$\int_U (f \circ \sigma)(x) D_\sigma(x) dx = \int_V f(t) n_\sigma(t) dt.$$

Proof. U is a disjoint union of sets $U_1, \ldots, U_t$ in $\Omega(K^n)$ on which σ is a homeomorphism. Hence, by Theorem (4.8) we have for each i

$$\int_{U_i} (f \circ \sigma)(x) D_\sigma(x) dx = \int_{\sigma(U_i)} f(t) dt.$$

Since $n_\sigma(t)$ is equal to the number of sets $\sigma(U_i)$ for which $t \in U_i$ we have

$$\int_U (f \circ \sigma)(x) D_\sigma(x) dx = \sum_i \int_{\sigma(U_i)} f(t) dt = \int_V f(t) n_\sigma(t) dt.$$

Remark (4.10). One can derive a chain-rule for m-differentiable functions. Moreover, one can show that if σ is a homeomorphism and D_σ exists, then σ^{-1} is also m-differentiable and its derivative at the point $\sigma(a)$ is equal to $D_\sigma(a)^{-1}$. We will leave the proofs to the reader.

REFERENCES

1. GÜNTZER, U., Zur Funktionentheorie einer Veränderlichen über einem vollständigen nichtarchimedischen Grundkörper, Arch. Math. **17**, 415–431 (1966).
2. KOKSMA, J. F. and C. G. LEKKERKERKER, Colloquium P-adische getallen, Mathematisch Centrum, Amsterdam.
3. MONNA, A., Sur les espaces linéaires normés V, Indag. Math. **10**, 197–210 (1948).
4. ———— and T. A. SPRINGER, Intégration non-archimédienne I, II, Indag. Math. **25**, 634–653 (1963).
5. SERRE, J-P., Classification des variétés analytiques p-adiques compactes, Topology, **3**, 409–412 (1965).
6. SCHIKHOF, W., Non-Archimedean Harmonic Analysis, (Thesis) Nijmegen, 1967.
7. BRUHAT, F., Sur les représentations des groupes classiques p-adiques I, Am. J. Math. **83**, 321–338 (1961).